(1)

DTIC FILE COPY

CORNELL AERONAUTICAL LABORATORY, INC.
Buffalo, New York

An Analysis of Track Mechanics to Improve the Simulation of the Ride Dynamics of Track-Laying Vehicles

Final Report

Report No. YM-1424-V-205

Contract No. DA-30-069-ORD-2913

OS Code 5510.12.21700.02

Document Control No. 010 095 041 041 1

July 1961

The work summarized in this report was performed under the technical supervision of the Dynamic Simulations Laboratory, Ordnance Tank-Automotive Command, Detroit Arsenal

Prepared by: Daniel C. Clark

Daniel C. Clark
Project Engineer
Vehicle Dynamics Dept.

Approved by: Fred Dell'Amico

Fred Dell'Amico, Head
Servo and Applied Mechanics Section
Vehicle Dynamics Dept.

Approved by: Leonard Segel

Leonard Segel
Assistant Head
Vehicle Dynamics Dept.

FOREWORD

This report was prepared by the Vehicle Dynamics Department of the Cornell Aeronautical Laboratory, Inc., Buffalo, New York, under research contract DA-30-069-ORD-2913 between the Department of the Army and CAL. It summarizes the work performed in an analysis of the influence of track mechanics on the simulation of the ride qualities of track-laying vehicles. This is the final report on that portion of the above contract dealing with track tension effects on vehicle ride behavior.

TABLE OF CONTENTS

NOMENCLATURE

Symbol	Description	Nominal Units
a	distance from CG to front axle	ft
b	distance from CG to rear axle	ft
s	undeflected length of idler pre-load spring	ft
h_1	undeflected length of front suspension spring	ft
h_2	undeflected length of rear suspension spring	ft
H_1	undeflected length of front bogey wheel spring	ft
H_2	undeflected length of rear bogey wheel spring	ft
L	total length of the track	ft
ℓ	free length of track between bogey wheels	ft
s_1	overhang from front axle to front idler wheel	ft
s_2	overhang from rear axle to rear idler wheel	ft
λ_1	track length from front idler wheel to front bogey wheel	ft
λ_2	track length from rear idler wheel to rear bogey wheel	ft
x	ground profile distance variable	ft
$z(x)$	ground profile function	ft
$\Delta(x)$	track profile function	ft
x_1	distance from origin of ground axes to the front axle	ft
x_2	distance from origin of ground axes to the rear axle	ft
Δ_1	track position relative to ground axes at $x = x_1$	ft
Δ_2	track position relative to ground axes at $x = x_2$	ft
y_1	position of front axle mass	ft
y_2	position of rear axle mass	ft
δ	position of hull CG	ft
$d\sigma$	element of track arc-length	ft
x_i	the value of x at the point where the track leaves the ground between bogey wheels	ft
θ	hull pitch angle	rad

Symbol	Description	Nominal Units
α_1	angle between the horizontal and the track extending toward the rear of the vehicle measured at the front bogey wheel	deg
α_2	angle between the horizontal and the track extending toward the front of the vehicle measured at the front bogey wheel	deg
β_1	angle from the horizontal to the section of track between the front bogey and idler wheels	deg
β_2	angle from the horizontal to the section of track between the rear bogey and idler wheels	deg
$\varphi(x)$	track slope angle	deg
φ_o	initial condition track slope angle	deg
$\Delta'(x)$	track slope	-
F_1	suspension force acting on the hull at the front of the vehicle	lb.
F_2	suspension force acting on the hull at the rear of the vehicle	lb.
$\bar{F}_1$	bogey wheel force acting on the track at the front of the vehicle	lb.
$\bar{F}_2$	bogey wheel force acting on the track at the rear of the vehicle	lb.
T	track tension	lb.
$P(x)$	vertical ground pressure beneath the track	lb/ft
T_o	initial condition track tension	lb.
$u(t)$	vehicle forward velocity	ft/sec
v	track velocity of propagation	ft/sec
t	real time	sec
f_R	track resonant frequency	cps
k_1	front suspension spring rate	lb/ft
k_2	rear suspension spring rate	lb/ft

Symbol	Description	Nominal Units
K_1	front bogey wheel spring rate	lb/ft
K_2	rear bogey wheel spring rate	lb/ft
k	idler pre-load spring rate	lb/ft
C_1	front suspension damping	lb sec/ft
C_2	rear suspension damping	lb sec/ft
μ	track-ground friction coefficient	-
g	gravity acceleration	ft/sec^2
M_1	front bogey wheel axle mass	slug
M_2	rear bogey wheel axle mass	slug
M_3	hull mass	slug
ρ	track mass density	slug/ft
I	hull pitch inertia	slug ft^2

$$\cdot = \frac{d}{dt}$$

$$' = \frac{d}{dx}$$

SUMMARY

A mathematical model is developed to predict the influence of track tension on the vertical rides qualities of track-laying vehicles. The effect of track mass on the dynamic track motions is included in the analysis along with a set of postulates relating the track and ground profiles. These postulates are used to develop mathematical models for the distribution of track tension and ground pressure as a function of the track and ground profiles.

The track-mechanics postulates are applied in the development of a vehicle ride qualities model for a hypothetical vehicle, and it is shown that the developed method can be extended to the analysis of more complex and realistic vehicle models in a straightforward manner..

The requirement of digital computer programming and computation in order to compare results with those predicted by present-day "point contact" models and experimental data is shown. Conclusions are drawn relating to the ease of programming the track mechanics model for digital computation.

Three appendices are added to illustrate use of the model by presenting a static calculation of track ground pressure, track profile, and vehicle response to a sinusoidal ground profile.

I. INTRODUCTION

In the development of the existing methods for analyzing the ride motions of track layers, little attention has been given to a detailed analysis of the kinetic behavior of track motions in the vertical plane and the effect of these motions on the ride qualities of track-laying vehicles. A significant amount of analytical work has been done to predict the dynamic behavior of various hull-suspension systems through the use of mathematical models, but these models ignore the fact that the track is essentially a distributed parameter system that is capable of transmitting forces and deflections from one point to another, along the track, as a function of suspension and track properties, input ground profile, and vehicle speed. On the other hand, it is recognized that the present techniques which (1) consider individual bogey wheels to bear against the ground directly and (2) neglect the distribution of forces and deflections caused by track tension, are probably valid as long as the input profile is not sufficiently severe to cause elements of the track to leave the ground. It appeared, therefore, that a means of accounting for the effects of track mass, track tension, and other track-wheel and track-ground interactions would considerably advance the state-of-the-art in computing the ride response of track-laying vehicles.

The requirement for research in the field of track mechanics, as related to the vehicle ride problem, was recognized by the Ordnance Tank-Automotive-Command, who, in turn, contracted with the Cornell Aeronautical Laboratory to perform an analysis of track mechanics in order to improve the existing dynamic simulation of the ride motions of track-laying vehicles. Specifically, the study was performed to

(1) determine the static deflection profile of a track in response to a specified ground profile,

(2) investigate the dynamic behavior of a track, and

(3) examine various methods of modeling a track for computation purposes.

This report details the results obtained at CAL during investigations into the above-mentioned areas of interest.

II. PROPERTIES OF TRACK-LAYING VEHICLE SUSPENSIONS

Ordinary track-laying vehicle suspensions are similar to the suspensions used in road vehicles, with the exception that the bogey wheels are enclosed by an endless track of constant length rather than allowed to run directly on the ground surface (See Ref 1). The track is "laid" on the ground ahead of the front wheels and picked up behind the rear wheels in a continuous process such that all intermediate wheels run on the track. The bogey wheels transmit the ground reaction forces, as modified by track tension, into the hull of the vehicle by means of suspension springs and dampers. The suspension elements combine with the various masses and inertias of the vehicle to form a complex dynamic system that responds to the input ground profile in a manner determined by the vehicle speed, the suspension configuration, and the track properties.

In contrast with ordinary road vehicles, many bogey wheels are used in track-laying vehicle suspensions for the purpose of distributing the hull reactions as evenly as possible along the track and into the ground. These additional suspension elements complicate the suspension dynamics through the addition of more degrees-of-freedom, but do not represent a serious obstacle to an analysis of the system as long as sufficient electronic computing equipment is available. Further, the use of computing equipment allows important system nonlinearities such as bump stops and nonlinear dampers to be included on a quantitative basis.

When the ground profile is such that track pre-tension tends to be reduced, idler or compensation wheels bear against the track in such a manner as to maintain the tension. A large pre-tension is normally used to keep the track taut between bogey wheels in order to help distribute the load more evenly. The track drive torque is applied through a drive sprocket that can be mounted on the hull in the front or rear, depending on the particular design.

The track is generally composed of a series of plates that are hinged together at many points along the length of the track such that only very small bending moments can be generated. The length of each individual track plate is small relative to the over-all length of track.

III. TRACK DYNAMICS

As noted in the previous section, a track is composed of many plates that are hinged together at close intervals to reduce the bending moment that can be generated due to forces applied to the track from above (bogey wheels) and below (ground reactions). Although there are a finite number of track plates in any given track, their small dimensions and large numbers suggest that the track can be assumed to be a continuous band under tension. In a mathematical representation this assumption essentially replaces the finite difference equations that are required to describe the motions of a finite number of track plates of finite length by a single differential equation representing a track consisting of an infinite number of track plates, each of zero length (that is, a continuous band).

Assuming that the track is continuous and that its response does not depend upon track width, the dynamic behavior of the track can be described by the one-dimensional wave equation, subject to various space boundary conditions at the point of ground or bogey wheel contact, and initial conditions prevailing at an instant in time. The wave equation is derived in Reference 2, and can be written as follows:

$$\frac{\partial^2 \Delta}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 \Delta}{\partial x^2} \quad ; \quad \Delta = \Delta(x,t) \tag{1}$$

In equation (1), the factor T/ρ represents the square of the velocity of propagation of the transverse waves that may occur in the track. That is,

$$V = \sqrt{\frac{T}{\rho}} \tag{2}$$

It is reasonable to assume that wave motions will only occur in sections of track that are suspended off the ground between bogey wheels, or ground projections, or both. The longest unsupported section of track that is possible along the ground is a section of track that is suspended between two adjacent bogey wheels. If it is assumed that this bogey wheel spacing is given by ℓ , the fundamental or lowest frequency of vibration of the track between the oogey wheels is given by:

$$f_R = \frac{V}{2\ell} \tag{3}$$

Typical values of f_R for a steel track under tension are twenty cps and above. The vibration amplitudes tend to be small because of the large track tension. Since the hull pitch and bounce resonant frequencies are much lower than twenty cps (e.g., around one cps) it can be concluded that track dynamic effects should have little influence on vehicle ride qualities. In the remaining sections of this report, it is assumed that the effects of track vibration are negligible; that is, the track is massless and responds instantaneously to applied forces.

IV. POSTULATES OF THE TRACK MECHANICS ANALYSIS

Based on (1) the assumption that track mass is negligibly small, or, more exactly, that the ratio of track tension to mass density is sufficiently large to permit the effects of track mass on track dynamics and static track deflections, to be neglected, and (2) physical reasoning, it is possible to state the following postulates of the present analysis, a priori.

1. A section of track in tension will not depart from a straight line unless acted on by an external force (e. g., a bogey wheel or ground reaction force).
2. All track elements must always be above ground.
3. Ground reaction forces are always positive (the track is not fastened to the ground).
4. Track deflections are equal to ground profile deflections when the track is in contact with the ground and equal to bogey wheel suspension deflections at the bogey wheels.
5. A section of track always comes into ground contact or leaves ground contact tangent to the local ground profile (an exception or rather, special case, occurs when the local ground profile slope is not defined as, for example, with a rectangular bump. These cases are easily handled, however).
6. The track length is constant and independent of tension.
7. If a bogey wheel is supported off the ground by the track tension, the ground reaction is zero beneath the bogey wheel and the bogey wheel position is an unknown to be determined.
8. If a bogey wheel contacts the ground through the track, the bogey wheel support is supplied by both ground reactions and components of track tension. Here the bogey wheel position is known and the ground reaction force is a variable to be determined.

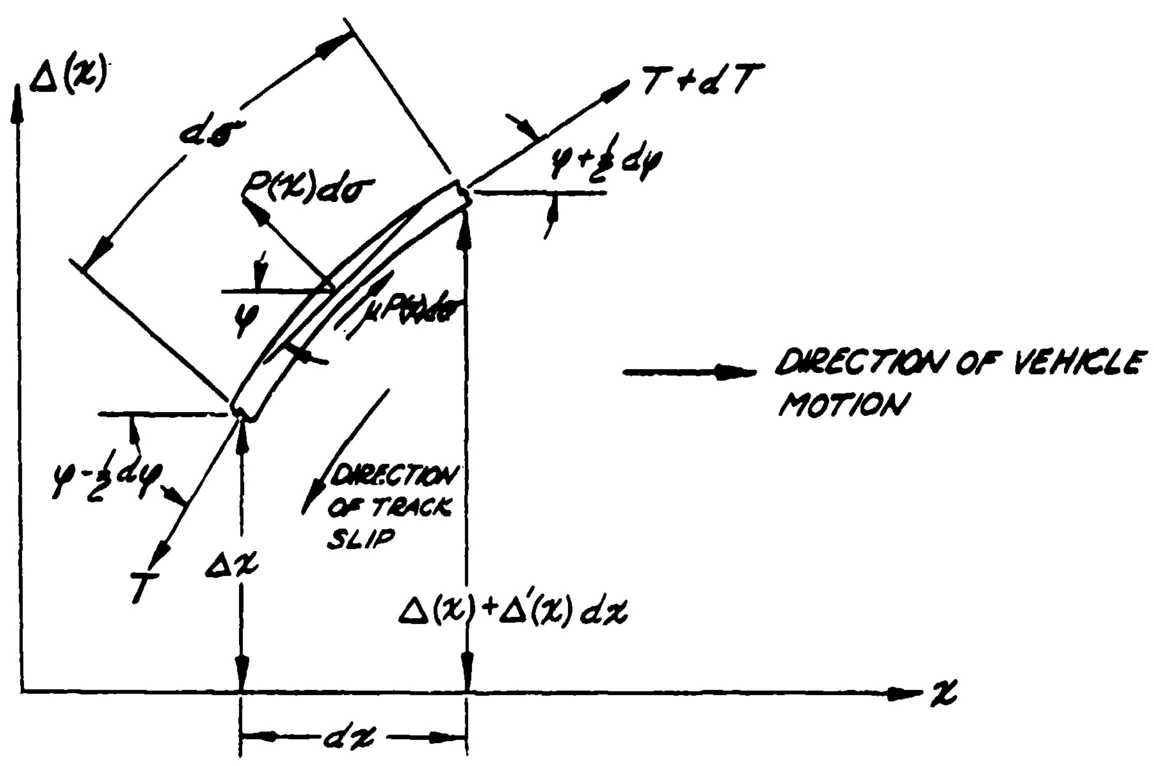
9. The length of arc of the track profile between the front and rear bogey wheels will always be the minimum distance possible with all track elements always on the ground or above the ground, for specified bogey wheel positions (assuming track tension cannot be zero).

These postulates, when combined with a mathematical vehicle model, allow the calculation of the force and moment distribution and the geometry of the vehicle and track profile at any given instant of time. The computation process is nonlinear and complex. It is described in subsequent sections of this report.

V. DISTRIBUTION OF TRACK TENSION AND GROUND PRESSURE

If a section of track between bogey wheels is caused to slide over the ground surface in the process of generating thrust, and a distributed normal pressure P(x) exists between the track and the ground, then the tension and ground pressure of the track will be distributed along the ground contact area in a manner prescribed by the track-ground friction, and the local ground profile slope and curvature.

Assuming the track to be a continuous band in static equilibrium, under tension, the forces that act on an element of track and the geometry of the local track (ground) profile can be combined to give the differential equation of the track tension distribution (see Figure 1).



AN ELEMENT OF FLEXIBLE TRACK SLIDING ON THE GROUND SURFACE UNDER THE ACTION OF TENSION AND NORMAL PRESSURE

Figure 1

Since the band is flexible and can support no bending moment, the summation of forces on an element of track in any two mutually orthogonal directions must be zero to insure static equilibrium. It is convenient to sum forces normal and tangent to the element in the "center" of the element (although any point on the element can be used since the arc length $d\sigma$ is infinitesimal in the limit). Upon performing these force summations, we get equations (4) and (5).

$$T \sin\left(\tfrac{1}{2}\, d\varphi\right) + (T + dT) \sin\left(\tfrac{1}{2}\, d\varphi\right) + P(x)\, d\sigma = 0 \tag{4}$$

$$- T \cos\left(\tfrac{1}{2}\, d\varphi\right) + (T + dT) \cos\left(\tfrac{1}{2}\, d\varphi\right) + \mu P(x)\, d\sigma = 0 \tag{5}$$

Recognizing that $\sin\left(\tfrac{1}{2}\, d\varphi\right) = \tfrac{1}{2}\, d\varphi$ and $\cos\left(\tfrac{1}{2}\, d\varphi\right) = 1$, equations (4) and (5) can be combined to give

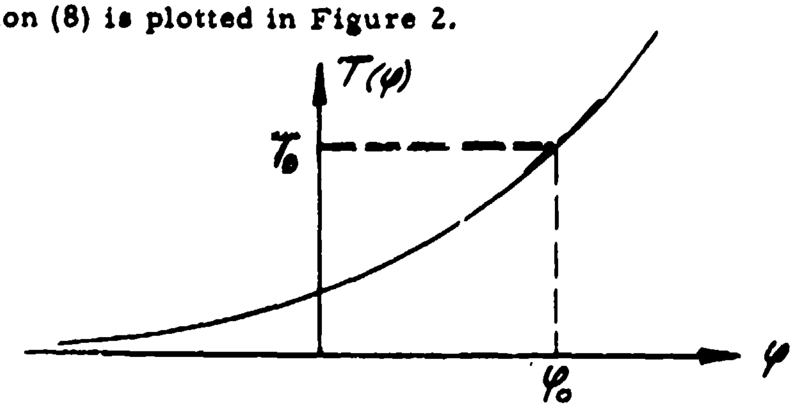
$$\frac{dT}{d\varphi} - \mu T = 0 \tag{6}$$

$$P(x) = - T(x) \frac{d\varphi(x)}{d\sigma} = - T(x) \frac{\Delta''(x)}{\left\{1 + \Delta'(x)^2\right\}^{3/2}} \tag{7}$$

It is evident that the distribution of tension as a function of ground slope angle can be determined from direct integration of equation (6), after evaluation of a single constant of integration. If $T(\varphi) = T_o$ when $\varphi = \varphi_o$,

$$T(\varphi) = T_o e^{\mu(\varphi - \varphi_o)} \tag{8}$$

Equation (8) is plotted in Figure 2.



DISTRIBUTION OF TENSION AS A FUNCTION OF LOCAL GROUND PROFILE SLOPE

Figure 2

Note that knowledge of the tension and ground profile slope angle at one point of the track along a section of track in ground contact (for ~ fixed friction coefficient) is sufficient to completely specify the distribution of tension with slope angle throughout the given ground contact range. The variation of ground profile slope angle with profile slope is obviously defined by equation (9).

$$\varphi(x) = \arc\tan \Delta'(x) \tag{9}$$

Consequently, the tension distribution as a function of distance along the ground from the displacement reference becomes:

$$T(x) = T_o e^{\mu \left\{ \arc\tan \Delta'(x) - \varphi_o \right\}} \tag{10}$$

$$\text{where} \quad \varphi_o = \arc\tan \Delta'(x_o) \tag{11}$$

For the sign conventions used in this analysis, the tension forces are greatest toward the rear of the vehicle. This is reasonable because the change in tension from front to rear of the vehicle, along the track, includes the vehicle drive thrust. The drive thrust is applied by pulling the track up and around the drive sprocket from the rear of the vehicle.

A numerical calculation of the tension and pressure distribution of a section of track sliding over a specified ground profile (a parabolic bump) is presented in Appendix A.

VI. A METHOD OF DETERMINING THE TRACK-GROUND PROFILE

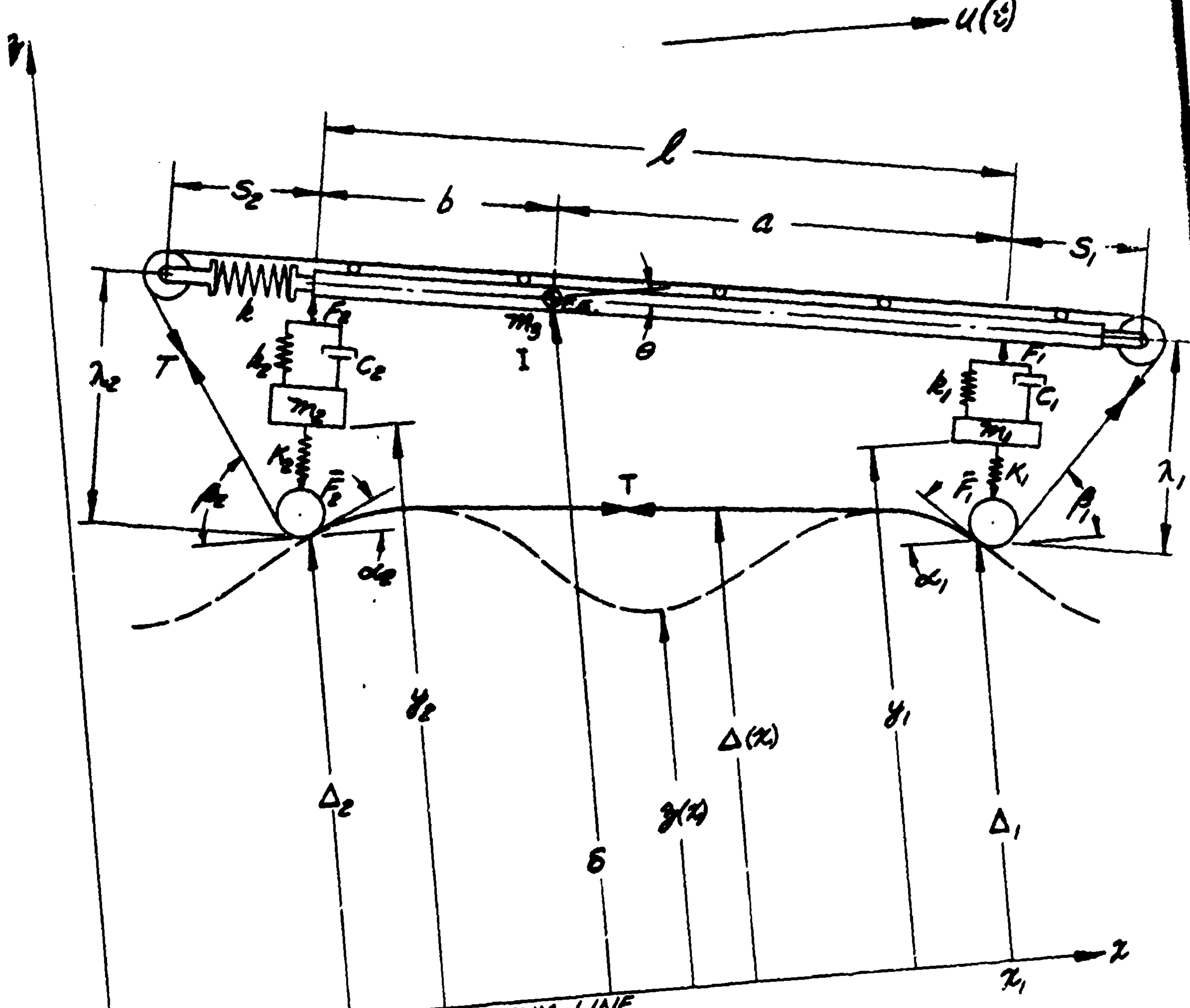
In order to compute the components of track tension that act on the various suspension elements of a given vehicle, the magnitude and direction of the tension vectors at each bogey wheel and the total length of track that lies between the front and rear bogey wheels must be known. The requirement of knowledge of tension vector directions and track arc length makes it essential that the track profile be determined as a part of the analysis. The track profile can be determined, for a given ground profile and set of bogey wheel positions, by application of the postulates of Section IV. It will be demonstrated in Section VII that iterative solution techniques are required for either static or dynamic calculations, and that the assumption of hull and bogey wheel positions (which are variables of the problem) represents the best place to start a given solution. The assumption of hull and bogey wheel positions is particularly convenient in that it allows the problem to be separated into two individual parts: the track properties and the hull response to the track force inputs. The solution mechanics are illustrated in Appendix B.

VII. APPLICATION OF THEORY TO A HYPOTHETICAL VEHICLE

It is possible to predict the static or dynamic response of a given tracked vehicle to a given terrain profile by the application of Newtonian equilibrium equations, and the postulates previously set down for the track mechanics to the analysis. In this section, two solution procedures (one static and one dynamic) will be outlined and applied to the hypothetical vehicle model shown in Figure 3. It is assumed that the reader has a knowledge of the Newton-Raphson iterative numerical analysis technique (used to solve systems of simultaneous nonlinear equations by a systematic trial-and-error process). A good description of the technique can be found in Reference 3 and a brief resumé is given in Appendix C.

The vehicle shown in Figure 3 is a two-dimensional representation of a tracked vehicle with four-degrees-of-freedom and a continuous track. The model is not a representation of any existing vehicle but was selected to illustrate the solution technique in a relatively simple manner. The track is kept under tension by the pre-load spring (k) and idler wheel under all circumstances, even when the bogey wheels leave the road. In the following analysis, it is assumed that any change in track tension caused by track-ground friction forces is small with respect to the pre-load tension. In other words, the tension is assumed to be the same at all points along the track (i.e., uniform) but not necessarily constant (with time or distance of the vehicle from the origin). The tension level is a variable to be determined in the analysis. It is further assumed that the bogey wheel radii are sufficiently small such that the effect of wrapping of the track around the individual bogey wheels can be neglected. The validity of this assumption depends on track tension and the "stiffness" of the ground.

Referring to Figure 3, all vertical displacement variables are measured with respect to an arbitrary datum line such that absolute displacements are used rather than relative displacements. This is necessary because vehicle geometry is an essential feature in the determination of the ride behavior of track-laying vehicles. It is not sufficient to know the relative position between two points, one must know the location of the various vehicle elements with



HYPOTHETICAL VEHICLE MODEL

Figure 3

respect to earth axes as well. This added complexity is forced into the analysis by the nature of the track force - displacement mechanics

Based on the model illustrated in Figure 3, the equations that follow can be written to describe the vehicle response to the terrain profile input, assuming the pitch angle, θ, can be considered to be a small angle.

$$M_3\ddot{\delta} - F_1 - F_2 + T(\sin\beta_1 + \sin\beta_2) + M_3 g = 0 \tag{12}$$

$$I\ddot{\theta} + F_1 a - F_2 b - T\left[(a + s_1)\sin\beta_1 - (b + s_2)\sin\beta_2\right] = 0 \tag{13}$$

$$M_1(\ddot{y}_1 + g) + F_1 = \bar{F}_1 \tag{14}$$

$$M_2(\ddot{y}_2 + g) + F_2 = \bar{F}_2 \tag{15}$$

Equations (12) and (13) are the hull force and moment balance equations. The summation of forces acting on the unsprung masses M_1 and M_2 are given in equations (14) and (15), where F_1, F_2, $\bar{F}_1$, and $\bar{F}_2$ are defined as follows:

$$F_1 = k_1\left[h_1 - \left\{\delta - a\theta - y_1\right\}\right] - C_1\left[\dot{\delta} - a\dot{\theta} - \dot{y}_1\right] \tag{16}$$

$$F_2 = k_2\left[h_2 - \left\{\delta + b\theta - y_2\right\}\right] - C_2\left[\dot{\delta} + b\dot{\theta} - \dot{y}_2\right] \tag{17}$$

$$\bar{F}_1 = K_1\left[H_1 - (y_1 - \Delta_1)\right] \tag{18}$$

$$\bar{F}_2 = K_2\left[H_2 - (y_2 - \Delta_2)\right] \tag{19}$$

The symbols used in equations (16) through (19) are defined in the nomenclature list at the beginning of this report.

Several equations must be written to describe the kinematics of the assumed vehicle. For example, the length of track is constant, but composed of various distances that are variables of the problem. The total track length is given by:

$$L = a + b + s_1 + s_2 + \sqrt{\lambda_1^2 + s_1^2} + \sqrt{\lambda_2^2 + s_2^2} + \int_{x_2}^{x_1} \sqrt{1 + \Delta'(x)^2}\, dx \qquad (20)$$

In equation (20), the individual terms are evident from Figure 3. The integral represents the length of arc of the track profile between bogey wheels, along the ground, as indicated by the solid line in the diagram. The value of this integral is determined for a given ground profile as illustrated in Appendix B. Other, more obvious, kinematic relations are:

$$\lambda_1 \approx \delta - a\,\theta - \Delta_1 \qquad (21)$$

$$\lambda_2 \approx \delta + b\,\theta - \Delta_2 \qquad (22)$$

$$\beta_1 \approx \arctan\left(\frac{\lambda_1}{s_1}\right) \qquad (23)$$

$$\beta_2 \approx \arctan\left(\frac{\lambda_2}{s_2}\right) \qquad (24)$$

Equations (21) through 24) are based on the assumption that $\sin\theta = \theta$, $\cos\theta = 1$, and β_1 and $\beta_2 >> \theta$.

The idler wheel translates the force in the pre-load spring into track tension. The dimension s_2 obviously depends on track tension. The equation for track tension as a function of compression of the idler spring is:

$$T = k\,(s - s_2) \cdot \frac{1}{1 + \cos\beta_2} \qquad (25)$$

Postulate 5 of Section IV of this report dictates that the track must be tangent to the ground profile when the track contacts the ground or at any point where the track leaves the ground without being acted on by bogey wheel forces. As a consequence of the postulate, the angles α_1 and α_2 are defined as follows:

$$\alpha_1 = -\arctan \Delta'(x_1 -) \qquad (26)$$

$$\alpha_2 = \text{arc tan } \Delta' \; (x_2 +) \tag{27}$$

The angle α_1 is taken positive in the clockwise sense, α_2 is positive counter-clockwise (Figure 3). This explains the minus sign in equation (26). The arguments x_1 - and x_2 + indicate that the track profile slope Δ' (x) is to be measured slightly to the rear of the front bogey wheel in determining α_1 and slightly to the front of the rear bogey wheel in determining α_2. This is necessary to eliminate the problem in definition of slope caused by the possible rapid variation in track slope beneath a bogey wheel.

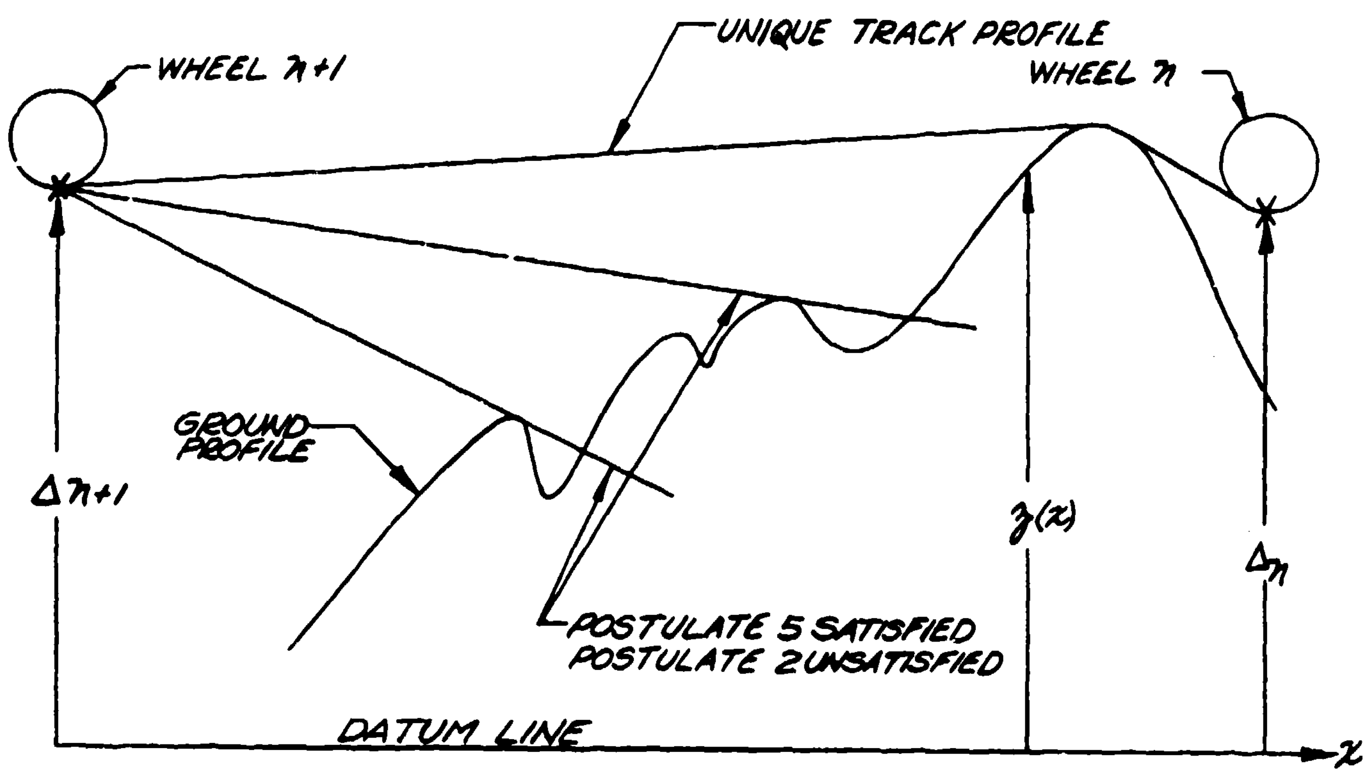
The lift forces on the bogey wheels caused by upward components of track tension are given by equation (28).

$$\bar{F}_n = T \, (\sin \alpha_n + \sin \beta_n) \qquad n = 1, \, 2 \text{ etc.} \tag{28}$$

This equation describes the total vertical bogey wheel force, for the n'th wheel, when the track beneath the bogey wheel is not in contact with the ground. If the track beneath the bogey wheel is in ground contact, additional bogey wheel lift forces occur due to the vertical component of the ground reaction force.

If a given bogey wheel is in contact with the ground through the track, the position of the bogey wheel is known, that is, $\Delta_n = z \, (x_n)$ for the n'th bogey wheel (postulate 8). For this case, equation (28) is not required to solve for α_n, since α_n can be determined from geometry alone. If, on the other hand, a given bogey wheel is suspended off of the ground by the lift components of track tension, the bogey wheel position Δ_n must be solved for before the track profile can be determined (postulate 7). Equation (28) must be used in the determination of the bogey wheel position for this case.

Once the bogey wheel position is known, the track profile can be calculated by a simultaneous solution of the linear track profile equation, and the ground profile equation, for the point where the track and ground profiles and slopes are equal (postulates 4 and 5). In the computation of track profile, it is also necessary that the computed track position must always result in all elements of the track being above ground level (postulate 2). These points are illustrated by Figure 4, when the location of bogey wheels n and n + 1 as well as the ground profile magnitude are assumed to be known. The bogey wheel positions are obvious only when the bogey wheels are on the ground. If they are off the ground, their positions must be determined from a simultaneous solution of the complete system of equations that describe the behavior of the vehicle. The ground profile is always an input quantity which is pre-specified.



TRACK PROFILE GEOMETRY

Figure 4

The track-profile requirements can be translated into mathematical terms for computation purposes in a variety of ways. For example, the ground profile could be divided into a number of intervals, and each interval fitted with an algebraic power series. It would then be possible to use analytic geometry to determine the point (s) of equal displacement and slope that satisfies the requirement that all parts of the track always stay above ground. The length of arc of the section of track along the ground could then be determined by piecewise integration of the various mathematical expressions for the track profile, in the various ranges where the track is either on the ground or above the ground. A sample profile computation is made in Appendix B.

a) Static Solution Technique

The above comments are valid regardless of whether a static or dynamic solution is to be obtained. There are two ways to determine the static response of a given vehicle to a given ground profile. One method is to obtain the steady-state response, at zero vehicle forward velocity, using a dynamic solution technique. With this method, the final solution values should be independent of the initial conditions assumed at the beginning of a solution. If the static solution is all that is desired, this method can be inefficient because of the time wasted in computing the unwanted transient response of the vehicle. It is more efficient to compute the static response directly, using a numerical analysis technique, such as the Newton-Raphson method. With this method, it is not necessary to compute any velocity or acceleration values.

The hypothetical vehicle (Figure 3) is a four-degree-of-freedom system (exclusive of the track). The spring mass can bounce and pitch, and each of the unsprung masses can bounce in the vertical plane. The Newton-Raphson method requires the assumption of numerical values for four of the independent system variables (e.g., δ, θ, Δ_1, and Δ_2). These values, termed "starting" values, can be obtained in many ways, but the usual method is to assume that the nonlinear terms in the system equations are negligible and solve the resulting simplified equations simultaneously for closed mathematical expressions. These expressions will generally give numerical starting values sufficiently close to the final answer to insure convergence of the iterative

Newton-Raphson method.

It is interesting to note that the number of unknowns to be computed depends on the position of any given bogey wheel relative to the ground. Since the ground profile is assumed to be known, the position of any bogey wheel in contact with the ground (through the track) is also known. For this case, the particular value of bogey wheel position does not have to be computed, and the complexity of computation is reduced. However, when a given bogey wheel is in contact with the ground, it is necessary to compute and monitor the vertical ground reaction force. This force must always be greater than or equal to zero. If it is greater than zero, the bogey wheel must necessarily be contacting the ground, resulting in a direct knowledge of bogey wheel position. If, however, the ground reaction force becomes zero, the bogey wheel no longer bears on the ground, and its position becomes a variable to be computed in the solution procedure. It is apparent that automatic computation by digital machines will require constant monitoring of the vertical ground reaction force beneath each bogey wheel and switching of the program to account for a changing number of variables as bogey wheels alternately contact or leave the ground.

The static analysis technique, using the Newton-Raphson method, is illustrated by a numerical example in Appendix C.

b) Dynamic Analysis Technique

The ultimate usage of the track mechanics theory is to effect a quantitative description of the ride motions of track-laying vehicles on irregular terrain. The inclusion of the word "ride" automatically requires that the analysis be extended to include time as an independent variable; that is, the dynamic vehicle motions are required.

The additional complexity, over the static analysis technique just described, introduced into the analysis by the addition of time-varying terms is not large from a purely technical or mathematical standpoint; however, the computer requirements are stringent, as explained below. Since the tracks can be assumed to respond instantaneously to external forces or tensions (Section III) they can be treated in the same way as in the static analysis previously described. The largest additional complication is the necessity of performing time integrations of the various acceleration and velocity terms to compute the transient positions of the various elements of the vehicle. For digital computations this would, for example, first require a computation of the instantaneous accelerations of the vehicle and its components by use of the Newton-Raphson method, and second require a computation of the change in velocity and position that results from these accelerations over a small time interval by use of a numerical integration technique such as the Runge-Kutta method. At each instant in time, both of these methods would be used. Since many time increments are required to approximate a continuous function (especially when the acceleration values are changing rapidly, as in the case of rough ride motions), the digital computer solution time for the dynamic vehicle response to a given terrain can be much greater than "real" time. There does not appear to be any way to circumvent this problem, because the track mechanics equations are too complex for analog computation. They require the generation of many nonlinear functions, the evaluation of definite integrals with independent variables other than time, and the solution of algebraic equations in space variables (with the boundary conditions dependent on computer "logic" tests). It is not possible to separate the track mechanics from the classical "point-contact" tire analyses, since the bogey wheel axle positions depend upon both hull motions and track tension effects in a closed-loop fashion. In other words, no equivalent ground profile "filtering" function can be derived to allow the use of a modified ground profile with the standard point-contact ride models now in existence.

For the hypothetical vehicle model of Figure 3, eight initial conditions are required to start a solution. The initial conditions required are four velocities and four positions, specified for the primary dependent variables of the vehicle (viz. δ, θ, y_1, and y_2). Eight initial conditions are required because of the four simultaneous second-order differential equations, (12) through (15), listed earlier in this section.

As in the static analysis case, it is necessary to monitor the vertical ground reaction beneath each individual bogey wheel so as to know when the wheel leaves the ground, and switch equations (28) into the analysis so that the bogey wheel position can be computed when it is off the ground.

VIII. REQUIREMENTS FOR FUTURE RESEARCH

The necessity for verification by experiment is an integral part of the development of any new theoretical description of a complex physical phenomenon. A requirement of the verification procedure in this case is to determine the additional "fidelity" of ride simulation afforded by the inclusion of distributed track tension in the analysis, as constrasted to the more conventional ride analysis methods where the track tension is neglected and the support, or bogey wheels are considered to bear directly against the ground at contact points.

In order to compare the ride response of a vehicle as given by the present analysis with that directly measured in the field and that predicted by current simplified (trackless) mathematical models, it is necessary that the dynamic equations of motion be programmed for solution on a digital computer. The theory is sufficiently complex, in terms of both numerical analysis techniques and logical decisions, as to make the use of large-scale, high-speed digital computing equipment very desirable. Once the computer program is operational, the ride response of a present-day tracked vehicle, to a given input ground profile, should be computed. The ride response of the same vehicle to the same ground profile should also (1) be computed with a conventional, trackless mathematical model and (2) measured in the field by experimental methods.

The degree of correlation between the two analytical methods and the measured results will be a direct indication of the necessity or desirability of including track tension in future analytical studies. It is important to note that the ground profile selected as the input for the above-mentioned test should be severe enough to cause bogey wheels to leave the ground, because the results predicted by the point-contact theory and the track tension theory should be almost the same for mild inputs with long wave lengths.

It is estimated that a digital computer program to solve the dynamic equations of ride motion in the xz plane (no roll or yaw freedom) will be sufficiently realistic to determine the utility of the theory of track-mechanics developed herein. Such a program would utilize a computer of the IBM-704 class and result in machine time calculations in the order of from ten to one hundred times "real" time. Smaller machines could be used with a corresponding increase in computing times.

IX. CONCLUSIONS

It can be concluded from the work described in this report that the amoung by which the ride behavior of track-laying vehicles is affected by tension in the tracks of the vehicle can be predicted by mathematical analysis. The amount of influence on vehicle ride qualities depends upon the magnitude of the track tension, the suspension properties, and the form of the ground profile. The order of magnitude of these track tension effects is unknown and will remain so until detailed calculations can be performed.

In the derivation of the track tension model, a set of track mechanics postulates was formed. Some of the postulates are obviously valid; others are subject to experimental verification.

The effects of track mass are negligible because the track flexural vibration frequencies are theoretically much higher than the resonant frequencies involved in the suspension and hull response and the amplitudes are correspondingly much lower. This conclusion is also subject to experimental verification.

A computing procedure has been hypothesized that can utilize a medium high-speed digital computer and should result in digital computations from ten to one hundred times slower than "real" time on such a machine.

The developed equations should be generalized and programmed for digital computations in order to facilitate a comparison of results yielded by this more refined model with those yielded by a point-contact model and experimental results obtained for the same vehicle operating on the same terrain.

The track mechanics model developed herein is a "closed-loop" model; that is, it cannot be separated from the suspension and hull response equations and represented as, for example, a "transfer function" to accept the true ground profile input and modify it to represent an equivalent ground input to the simple point-contact models in use today. This closed-loop nature of the track mechanics model is caused by the necessity of knowing bogey wheel axle positions in order to compute the track tension effects; the bogey wheel positions cannot be computed without simultaneous knowledge of both track and hull inputs to the axle, thereby precluding a pre-calculation of the effect of track tension upon the distribution of the ground profile into the suspension.

The assumptions of only two bogey wheels, uniform track tension, and small hull pitch angles made in Section VII, are not basic to the analysis. These assumptions were made for simplicity in describing the methods developed and would be discarded if detailed calculations were made to determine the large amplitude motions of a vehicle in which the track is suspended on many bogey wheels. The computational methods would remain unchanged.

X. REFERENCES

1. Sattinger, I. J., et. al., "Analysis of the Suspension System of the M-47 Tank by Means of Simulation Techniques", Report No. 2023-2-T, The University of Michigan, June 1954.

2. Thompson, W. T., "Laplace Transformation", Prentice-Hall, New York, 1950.

3. Margenau, H. and Murphy, G. M., "The Mathematics of Physics and Chemistry", D. Van Nostrand Co., New York, 1943, pp 475-477.

APPENDIX A

TRACK TENSION AND PRESSURE DISTRIBUTION

In this section, a numerical calculation of the tension and pressure distribution of a section of track slipping on a parabolic ground profile is performed. The track-ground friction coefficient is arbitrarily assumed to be unity.

Assume that the ground profile can be represented by equation A-1.

$$z(x) = -\tfrac{1}{2}\,x^2 \qquad \text{(A-1)}$$

If we further assume that the track is wrapped symmetrically around the parabolic bump out to $x = \pm 2$, as shown in Figure A1a, we can write the following equations:

$$\left.\begin{aligned} \Delta(x) &= -\tfrac{1}{2}\,x^2 \\ \Delta(x) &= -x \\ \Delta(x) &= -1 \end{aligned}\right\} \quad -2 \lessgtr x \lessgtr 2 \text{ ————————————— (A-2)}$$

From Section V, equations (11), (10), and (7), we can substitute the numerical values obtained from Figure A1a and get:

$$\varphi_0 = \text{arc tan } \Delta'(x_0) = \text{arc tan } (-x_0) = \text{arc tan } 2 = 63.44 \text{ deg.}$$

$$\frac{T(x)}{T_0} = e^{\mu\left\{\text{arc tan } \Delta'(x) - \varphi_0\right\}} = e^{1\left\{\text{arc tan } (-x) - 63.44°\right\}}$$

$$\frac{T(x)}{T_0} = e^{-\left\{\text{arc tan } x \;+\; 63.44°\right\}} \qquad \text{(A-3)}$$

$$\frac{P(x)}{T_0} = -\frac{\Delta''(x)}{\left\{1 + \Delta'(x)^2\right\}^{3/2}} \cdot \frac{T(x)}{T_0} = \frac{1}{\left(1 + x^2\right)^{3/2}} \cdot \frac{T(x)}{T_0} \qquad \text{(A-4)}$$

Equations (A-3) and (A-4) are plotted in Figure A1b and A1c. Note that the tension decreases 89 percent across the parabola with the assumed large slope angle (ψ_o) and friction coefficient. This presents a strong argument for maintaining large values of pre-tension in order to minimize the length of track in contact with the ground in the vicinity of a bump. The converse is true with regard to the ground pressure beneath the track. Since the (shaded) area beneath the curve of Figure A1c represents the total vertical ground reaction for the parabolic bump, it is desirable to have large slope angles such that the vertical load can be distributed over as much of the ground and track as possible and the local ground and track stresses will be minimized. It is also apparent that ride behavior can suffer on rough terrain when the pre-tension is reduced because of the increased axle displacements that occur as the track envelopes the bump. In the extreme, with zero track tension, the vehicle will behave like a wheeled, trackless, vehicle.

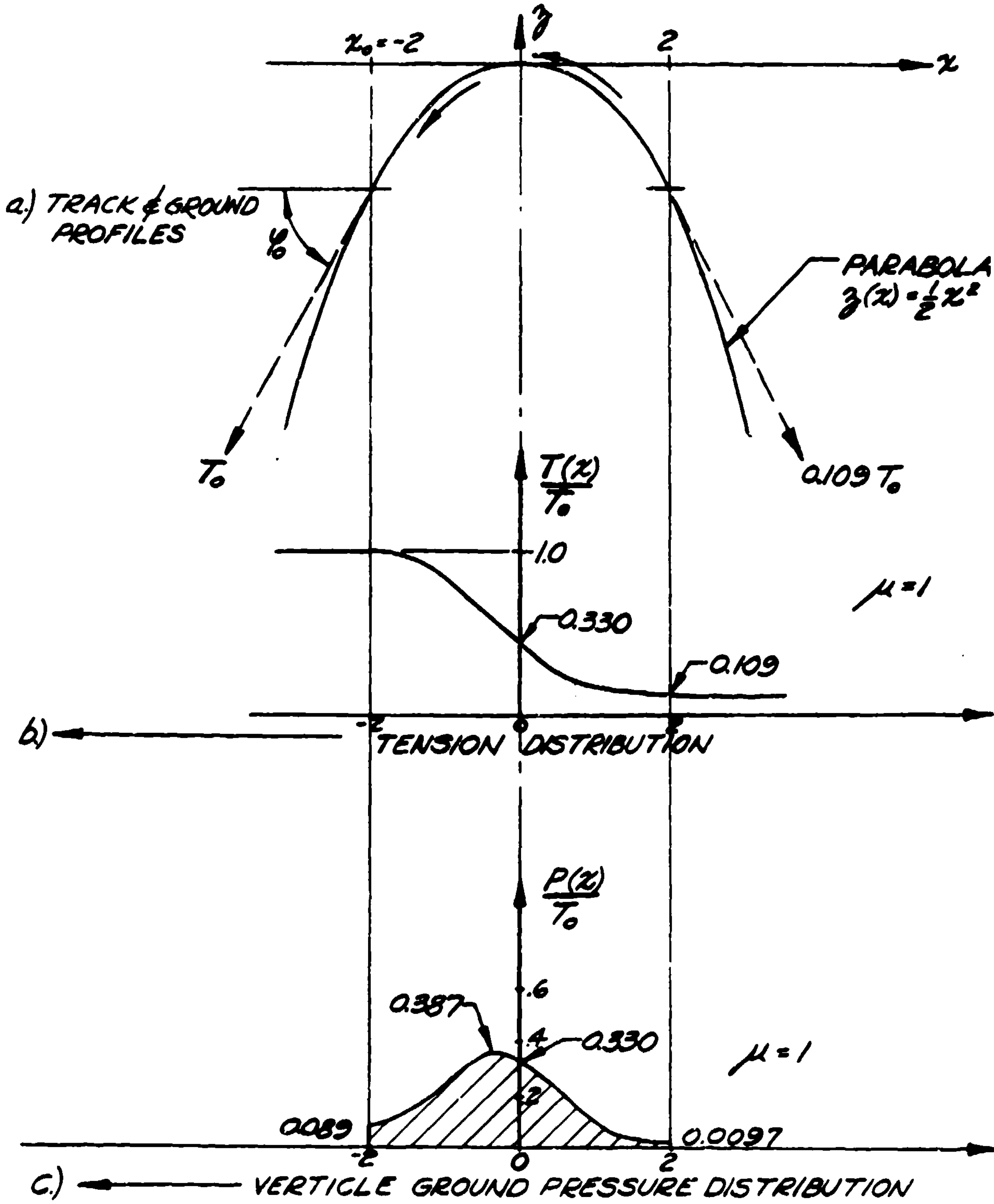


FIGURE A1 TRACK TENSION AND PRESSURE DISTRIBUTION

APPENDIX B
TRACK-GROUND PROFILE DETERMINATION

As described in Sections VI and VII of the text, it is necessary to compute the track profile at every instant of time, in order to insure that the length of the track remains constant. If, for example, an extremely "sharp" bump is encountered, the track will tend to envelop it. This will cause the length of track running along the ground to increase and the length of track that runs over the pre-load idler wheel to the front of the vehicle to decrease. It is apparent that while total track length will be held constant, the suspension forces and displacements will change, thus requiring a knowledge of the distribution of track length. In this section, a hypothetical ground profile and vehicle is assumed, and the track profile is computed.

Suppose, for example, that the ground profile is given by:

$$z(x) = 1 + \sin x \tag{B-1}$$

Also assume that the front and rear bogey wheels of a two axle vehicle (see Figure C-1) bear on the ground at $x = x_2$ for the rear axle and $x = x_1$ for the front axle, as shown in Figure B1.

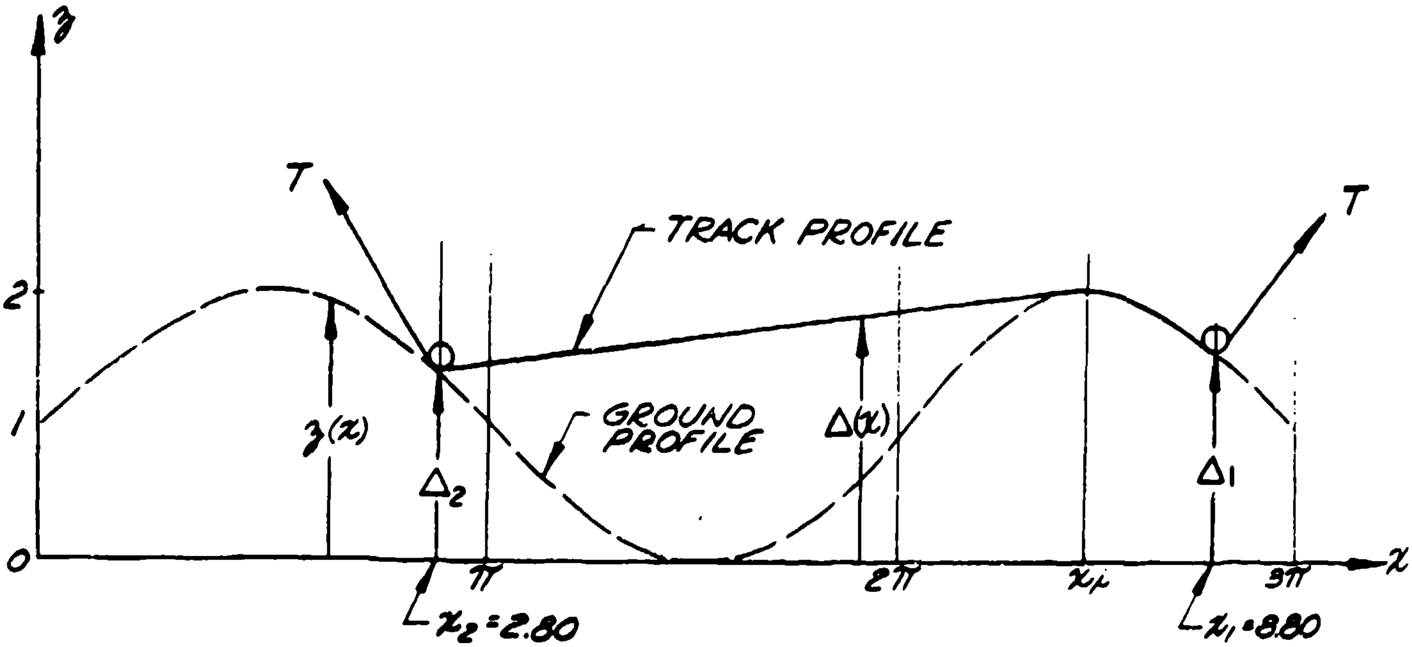


FIGURE B1 ASSUMED GROUND AND TRACK PROFILE

Since the track must contact the ground tangent to the local ground profile at point x_i, according to postulates 4 and 5, Section IV, we can wr'te:

$$z(x_i) = \Delta(x_i) = 1 + \sin x_i \qquad \text{(B-2)}$$

$$z'(x_i) = \Delta'(x_i) = \cos x_i \qquad \text{(B-3)}$$

The track is off the ground in the interval $x_2 \leq x \leq x_i$ and, according to postulate 1, can be represented by equation (B-4).

$$\Delta(x) = \Delta_2 + \Delta'(x_i)[x - x_2] \; ; \; x_2 \leq x \leq x_i \qquad \text{(B-4)}$$

where $\Delta_2 = 1 + \sin x_2$

In the interval $x_i \leq x \leq x_1$ the track is in contact with the ground and $z(x) = \Delta(x)$. It remains to find the point of ground contact at $x = x_i$. To do this, we set equation (B-2) equal to equation (B-4) evaluated at $x = x_i$, that is:

$$\Delta(x_i) = 1 + \sin x_i = (1 + \sin x_2) + \Delta'(x_i)[x_i - x_2] \qquad \text{(B-5)}$$

If we combine equations (B-3) and (B-5), and simplify, equation (B-6) results.

$$\sin x_i - \sin x_2 = (x_i - x_2) \cos x_i \qquad \text{(B-6)}$$

The values of x_i that satisfy equation (B-6) can be determined by numerical analysis after the location of the vehicle on the ground is specified. For example, if $x_2 = 2.800$ and $x_1 = 7.800$, as indicated on Figure B1, the two sides of equation (B-6) can be plotted to show the solution points; that is, the points where the two curves intersect. Figure B2 shows such a plot and indicates that the correct solution value is slightly less than $\frac{5}{2}\pi$.

A more accurate solution is obtained by using this point as a starting value and solving the equation by Newton's method. The more accurate solution is $x_i = 7.720$. The track profile then becomes:

$$\left.\begin{aligned} \Delta(x) &= 0.1333\,(x - 2.800) + 1.335; \quad 2.800 \leq x \leq 7.720 \\ \Delta(x) &= 1 + \sin x \quad ; \quad 7.720 \leq x \leq 8.800 \end{aligned}\right\} \qquad \text{(B-7)}$$

After the track profile is specified, the length of track along the ground, beneath the vehicle, must be computed. This length is represented by equation (B-8).

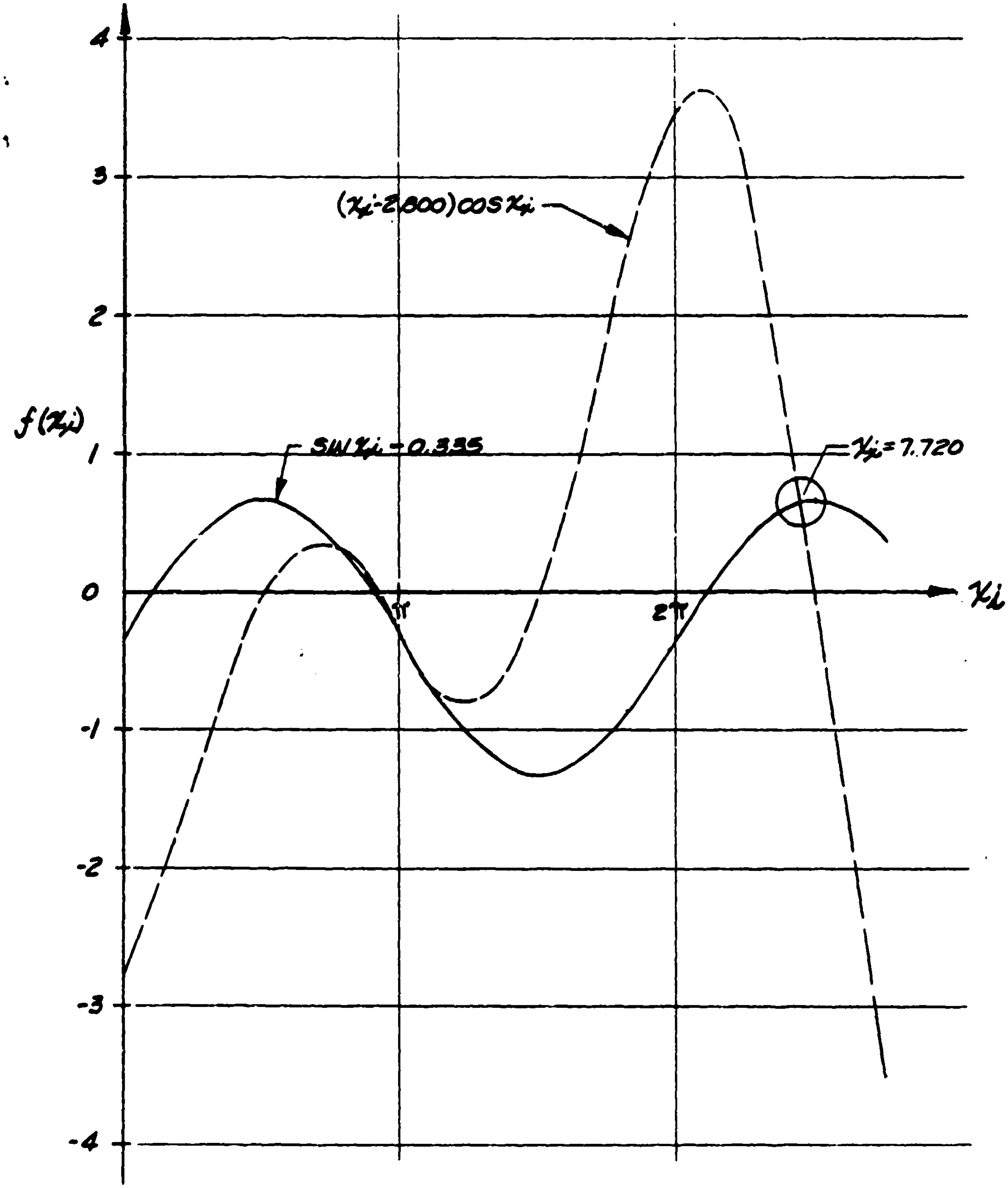


FIGURE B2 SOLUTION OF EQUATION B-6
FOR x_2 = 2.800

$$J = \int_{x_2}^{x_1} \sqrt{1+\Delta'(x)^2}\, dx \qquad \text{(B-8)}$$

It is apparent that equation (B-8) must be split into two integrals for the two track profile equations. When the numerical values are substituted, we get:

$$J = \int_{2800}^{7.720} \sqrt{1+(0.1333)^2}\, dx + \int_{7.720}^{8.800} \sqrt{1+\cos^2 x}\, dx = 4.964 + 1.188$$

$$J = 6.152$$

The second integral above is an elliptic integral of the second kind, and is a tabulated function. The above value of 1.188 was also obtained by the use of Simpson's one-third rule, which is particularly suited to digital computations, and would necessarily have to be used for more complex ground profiles (or at least some form of numerical integration will be necessary for most profiles).

APPENDIX C

THE STATIC RESPONSE OF A HYPOTHETICAL VEHICLE TO AN ASSUMED GROUND PROFILE

In this Section, we will illustrate the static suspension response of a hypothetical vehicle, to the ground profile illustrated in Appendix B, by a numerical calculation.

Assume that the vehicle characteristics are as represented in Figure C1. Since we are performing a static calculation only, the axle masses are assumed to be negligible, and the bogey wheel and suspension springs are lumped together with combined spring rates of k_1 and k_2. We will first write the equations of static equilibrium, viz:

$$Mg - F_1 - F_2 + T(\sin\beta_1 + \sin\beta_2) = 0$$

$$aF_1 - bF_2 - T\left\{(a + s_1)\sin\beta_1 - (b + s_2)\sin\beta_2\right\} = 0$$

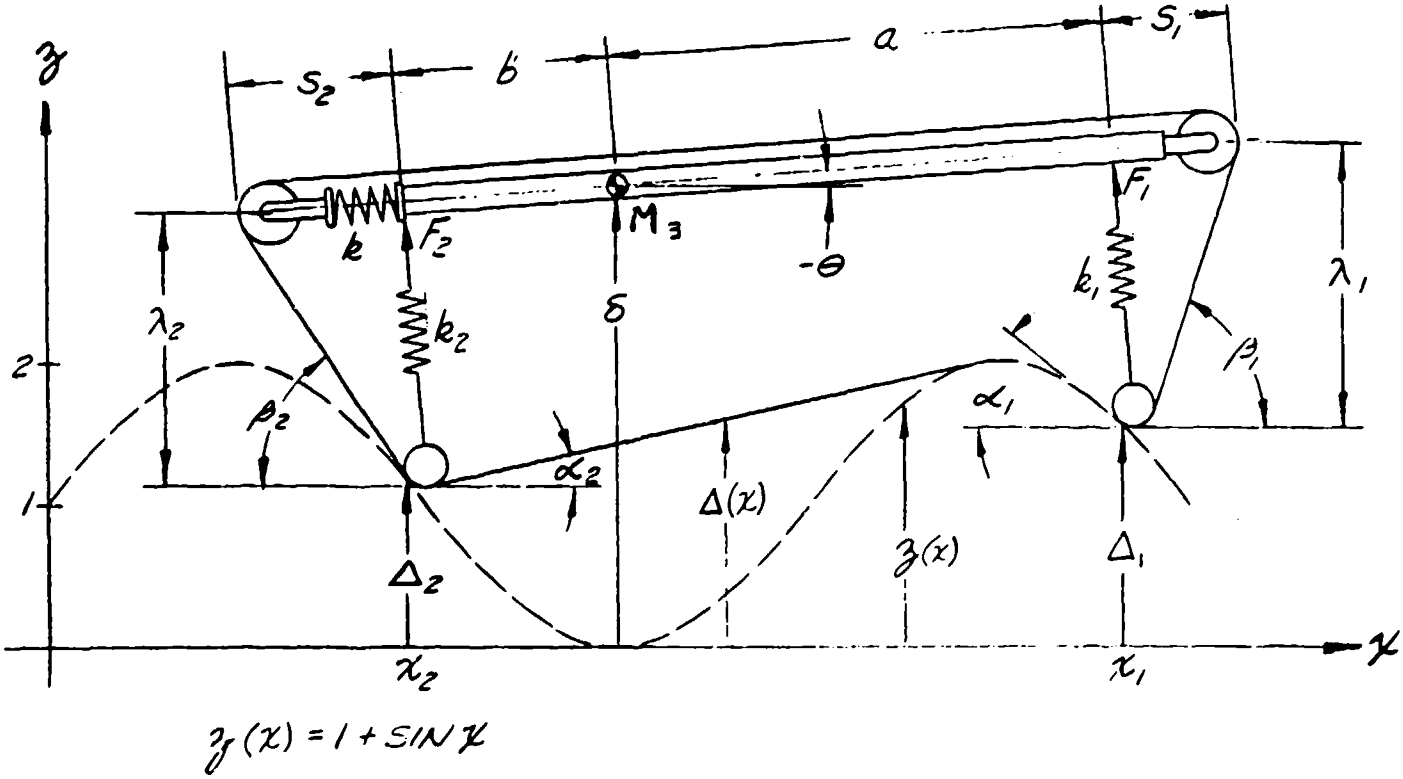


FIGURE C1 ASSUMED VEHICLE MODEL FOR NUMERICAL EXAMPLE - APPENDIX C

$$F_1 = k_1 (h_1 - \lambda_1)$$

$$F_2 = k_2 (h_2 - \lambda_2)$$

$$T = \frac{(s - s_2) k}{1 + \cos\beta_2}$$

Evaluation of the kinematic restraints on the system results in the following additional equations:

$$J = \int_{x_2}^{x_1} \sqrt{1 + \Delta'(x)^2}\, dx$$

$$\lambda_1 = \delta - a\theta - \Delta_1$$

$$\lambda_2 = \delta + b\theta - \Delta_2$$

$$\beta_1 = \text{ARC TAN} \left(\frac{\lambda_1}{S_1}\right)$$

$$\beta_2 = \text{ARC TAN} \left(\frac{\lambda_2}{S_2}\right)$$

$$L = a + b + S_1 + S_2 + \sqrt{\lambda_1^2 + S_1^2} + \sqrt{\lambda_2^2 + S_2^2} + J$$

Before we can solve the equations written so far, it is necessary to specify the ground profile and the location of the vehicle on the ground. This has been done, for this example, in Appendix B. The resulting ground profile is represented by equation (B-7) and the length of arc is shown to be $J = 6.152$. These equations and calculations require both bogey wheels to be on the ground. The remaining unknowns are: β_1, β_2, λ_1, λ_2, S_2, δ, θ, F_1, F_2, and T. The reader will note that there are ten independent equations above that must be solved simultaneously to result in numerical values for these ten variables. It is necessary to assume values for the parameters before a solution can be obtained. The following values were selected:

$a = 4$ ft	$h_1 = 2$ ft	$k = k_1 = k_2 = 1000$ lb/ft
$b = 2$ ft	$h_2 = 2$ ft	$M_3 = 62.11$ slugs
$s_1 = 1$ ft	$L = 16.5$ ft	$s = 1$ ft

Only in very special cases is it possible to solve a system of nonlinear equations in closed mathematical form. The above set of equations cannot be solved for the individual variables without recourse to an iterative, or trial-and-error solution technique. One such technique that has been used with success is the Newton-Raphson method. In this method, initial guess values are selected for at least as many variables as there are degrees-of-freedom in the system. These guess values are used to compute the numerical values of the functions and certain of their first derivatives. These numerical values are combined to result in correction increments, which, when added to the initial guess values, result in new guess values that are closer to the true solution. If the process is continued, any desired degree of accuracy can be obtained. Convergence of the method depends somewhat upon the accuracy of the initial guess values and the magnitude of the derivatives (e.g. zero values of the derivatives cause considerable trouble). In general, however, physical systems can be mathematically modeled such that a set of starting value equations can be derived by "linearizing" the model equations; that is, by neglecting all nonlinear terms. If solutions are required in the range where the nonlinear effects are dominant, starting values can be obtained by using previous solution values for the starting values of the next case and working up from the more linear range. This method will always be successful provided that a sufficiently small increment in variables is assumed, and that a solution exists. Reference 3 gives a good quantitative description of the application of the Newton-Raphson method to multi-variable systems. This method was used to solve the above system of equations for the static response of the vehicle to the assumed ground profile input. The solutions are given below:

$$\left.\begin{aligned} \Delta_1 &= 1.585 \text{ ft.} \\ \Delta_2 &= 1.335 \text{ ft.} \\ \alpha_1 &= 39.04 \text{ deg.} \\ \alpha_2 &= 7.59 \text{ deg.} \\ J &= 6.152 \text{ ft.} \end{aligned}\right\}$$

Input quantities: Determined in closed form in Appendix B and from equations (26) and (27), Section VII.

$$\left.\begin{aligned} \lambda_1 &= 1.732 \text{ ft.} & \delta &= 2.227 \text{ ft.} \\ \lambda &= 0.5979 \text{ ft.} & \theta &= -8.44 \text{ deg.} \\ \beta_1 &= 50.93 \text{ deg.} & F_1 &= 768.4 \text{ lb.} \\ \beta_2 &= 37.50 \text{ deg.} & F_2 &= 1402 \text{ lb.} \\ S_2 &= 0.7793 \text{ ft.} & T &= 123.1 \text{ lb.} \end{aligned}\right\}$$

Output Quantities: Newton-Raphson Solution Values